IV) $SO(4)$-bundles are isomorphic
$\Leftrightarrow$
w_2, p_1, e agree

exercise: prove the above
I) "easy" II) "easyish"
III), IV) harder

B. Characteristic classes

another way to think of Steifel-Whitney classes

Thm 5:

$\exists$ a unique function

$$w_i : Vect(M) \to H^i(M; \mathbb{Z}/2) \quad \forall M$$

satisfying

1) $w_i(f^*E) = f^* w_i(E) \quad \forall f : M \to N$
2) $w_0(E) = 1, \; w_i(E) = 0 \quad \forall i >$ fiber dim E
3) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$
   where $w(E_i) = 1 + w_1(E_i) + w_2(E_i) + \ldots$
4) $w_1(\gamma_n) \neq 0$ where γ_n is the universal line bundle over $\mathbb{R}P^n$

for 3) $E_1 \oplus E_2$ is called the direct sum of E_1 and E_2 and has fiber the direct sum of the fibers of E_1 and E_2

formally it is defined as follows

given $\begin{array}{ccc} \mathbb{R}^n \to & E_1 \\ & \downarrow \\ & M \end{array}$ and $\begin{array}{ccc} \mathbb{R}^m \to & E_2 \\ & \downarrow \\ & N \end{array}$

we cleanly get $\begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^m \to & E_1 \times E_2 \\ & \downarrow \\ & M \times N \end{array}$

if $M = N$ and $\Delta : M \to M \times M : m \mapsto (m, m)$

then $E_1 \oplus E_2 = \Delta^*(E_1 \times E_2)$

exercise: if $\{U_\alpha\}$ is a cover of M giving local trivializations of E_1 and E_2 with transition functions $\{\tau_{\alpha\beta}\}$ and $\{\tau'_{\alpha,\beta}\}$ then the transition functions for $E_1 \oplus E_2$ are

$$\tau_{\alpha\beta} \oplus \tau'_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n+m; \mathbb{R})$$
$$x \longmapsto \begin{pmatrix} \tau_{\alpha\beta}(x) & 0 \\ 0 & \tau'_{\alpha\beta}(x) \end{pmatrix}$$

for 4) recall $\gamma_n = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in \ell\}$

exercise: γ_n is a line bundle over $\mathbb{R}P^n$

4) $\Rightarrow$ $w_1(\gamma_n)$ generates $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ for all n

note: $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2 \hookrightarrow \cdots \hookrightarrow \mathbb{R}P^n$

let $\mathbb{R}P^\infty = \varinjlim_n \mathbb{R}P^n = \bigcup_n \mathbb{R}P^n$

we also get γ over $\mathbb{R}P^\infty$ and 4) $\Rightarrow w_1(\gamma) \neq 0$

exercise:

1) using $i: \mathbb{R}P^n \hookrightarrow \mathbb{R}P^m$ we have

$$i^*(\gamma_m) = \gamma_n$$

so if $w_1(\gamma_1) \neq 0$ then true $\forall \gamma_n$

2) show $\mathbb{R}P^1 = S^1$ and γ_1 = infinite Möbius band

$$\begin{array}{ccc} \mathbb{R} \to & \gamma_1 & \\ & \downarrow & \\ & S^1 & \end{array}$$ is non-orientable bundle

so from above $w_1(\gamma_1) \neq 0$

note: We have already shown w_i satisfies 1), 2), 4)

so to prove theorem just need 3) & uniqueness

before we do this let's consider some consequences!

easy consequences:

1) if E_1 and E_2 are isomorphic then

$$w_i(E_1) = w_i(E_2) \quad \forall i$$

(by property 1))

2) if E is a trivial bundle then

$$w_i(E) = 0 \quad \forall i > 0$$

(this follows from obstruction definition, but also from 1) since if $E \to M$ trivial, then let $f: M \to \{x_0\}$ and we have $E = f^*\{x_0 \times \mathbb{R}^n\}$

so $w_i(E) = f^* w_i(x_0 \times \mathbb{R}^n) = 0 \quad \forall i > 0$)

3) if E' is trivial and E is any bundle then

$$w_i(E \oplus E') = w_i(E)$$

Recall Whitney proved that any n-manifold embeds in $\mathbb{R}^{2n}$ and immerses in $\mathbb{R}^{2n-1}$

Th$^{\underline{m}}$ 6: $\boxed{\text{If } \mathbb{RP}^{2^r} \text{ can be immersed in } \mathbb{R}^{2^r+k}, \text{ then } k \text{ must be at least } 2^r - 1}$

so Whitney's th$^{\underline{m}}$ can't be improved for all manifolds!

note: If $f: M^n \to \mathbb{R}^k$ is an embedding, then we have the normal bundle $\nu(M)$ to $f(M)$

$$\nu(M) = \{ v \in T_x\mathbb{R}^k \mid v \perp T_x M, x \in M \}$$

and $TM \oplus \nu(M) \cong T\mathbb{R}^k|_M = \underbrace{M \times \mathbb{R}^k}_{\text{trivial bundle}}$

exercise: Show $\nu(M)$ well-defined if f just an immersion and still have

$$TM \oplus \nu(M) = f^* T\mathbb{R}^k = M \times \mathbb{R}^k$$

so to prove theorem we first study the S-W classes of E and E' s.t. $E \oplus E' =$ trivial bundle

so $w(E) \cup w(E') = w(\text{trivial}) = 1$

$$\therefore \boxed{w(E') = (w(E))^{-1}}$$

$$= \left(1 + (w_1(E) + w_2(E) + \dots)\right)^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 - \dots$

$$= \left(1 - (w_1(E) + w_2(E) + \dots) + (w_1(E) + w_2(E) + \dots)^2 - (w_1(E) + w_2(E) + \dots)^3 - \dots\right)$$

$$= 1 \underbrace{- w_1(E)}_{\text{order 1}} + \underbrace{(w_1^2(E) - w_2(E))}_{\text{order 2}}$$

$$+ \underbrace{(-w_1^3(E) + 2w_1(E) \cup w_2(E) - w_3(E)}_{\text{order 3}}) + \dots$$

so $w_1(E') = -w_1(E)$

$w_2(E') = w_1^2(E) - w_2(E)$

$w_3(E) = -w_1^3(E) + 2w_1(E) \cup w_2(E) - w_3(E)$

$\vdots$

<u>example</u>: $S^n \subset \mathbb{R}^{n+1}$ $\quad \nu(S^n) = S^n \times \mathbb{R}$

so $w(TS^n) = (w(\nu(S^n)))^{-1} = 1$

i.e. $w_i(TS^n) = 0 \;\; \forall \; i > 0$ $\quad$ we knew for $i \neq n$ since

$H^i(S^n)=0 \quad i\neq 0,n$

so can't distinguish TS^n from $S^n \times \mathbb{R}^n$ using S-W classes

(but note, e.g. TS^2 not trivial since it has no nonzero section)

recall $H^i(\mathbb{R}P^n;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & 0\le i\le n \\ 0 & 0 \end{cases}$

in fact $H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[a] \big/ \langle a^{n+1}=0\rangle$

polynomials in a w/ coeff in $\mathbb{Z}/2$ and $a^k=0$ for $k>n$

lemma 7:

$$\boxed{\begin{gathered} w(T\mathbb{R}P^n) = (1+a)^{n+1} = 1+\binom{n+1}{2}a+\binom{n+1}{2}a^2+\dots+\binom{n+1}{n}a^n \\ \text{in } H^*(\mathbb{R}P^n;\mathbb{Z}/2) \end{gathered}}$$

examples: $w(T\mathbb{R}P^2) = 1+a+a^2$

$w(T\mathbb{R}P^3) = 1 \qquad (1+4a+6a^2+4a^3)$

$w(T\mathbb{R}P^4) = 1+a+a^4$

Cor 8:

$w(T\mathbb{R}P^n)=1 \iff n+1$ is a power of 2
so if $n+1$ not a power of 2 then $T\mathbb{R}P^n$ is <u>not</u> trivial

Proof:

recall modulo 2 $(a+b)^2 = a^2+b^2$

$$\therefore (1+a)^{2^n} = 1+a^{2^n}$$

so if $(n+1) = 2^{2r}$ then

$$w(T\mathbb{R}P^n) = (1+a)^{n+1} = 1 + a^{n+1} = 1$$

if $n+1 = 2^r m$ with m odd $m>1$, then

$$w(T\mathbb{R}P^n) = (1+a)^{n+1} = \left((1+a)^{2^r}\right)^m$$

$$= (1+a^{2^r})^m \underset{\text{mod } 2}{=} 1 + m a^{2^r} + \frac{m(m-1)}{2} a^{2 \cdot 2^r} + \dots$$

$$\neq 1$$

to see last equality recall $\binom{n+1}{i}$ is given by Pascals triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
\quad \text{mod } 2 \text{ is} \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 0 & & 0 & & 0 & & 1 & \\
1 & & 1 & & 0 & & 0 & & 1 & & 0 \\
 & & & & & \vdots & & & & &
\end{array}$$

Remark: only $\mathbb{R}P^1, \mathbb{R}P^3, \mathbb{R}P^7$ are trivial

Proof of Th$\underline{\text{m}}$ 6 given lemma 7:

if n is a power of 2 then as above $\binom{n}{i}$ is

$$1 \; 0 \; \dots \; 0 \; 1$$

so $\binom{n+1}{i}$ is $1 \; 1 \; 0 \dots 0 \; 1 \; 1$

$$\therefore \; w(T\mathbb{R}P^n) = (1+a)^{n+1} = 1 + a + a^n$$

one may easily check

$$(w(T\mathbb{R}P^n))^{-1} = 1 + a + a^2 + \dots + a^{n-1}$$

thus if there is an immersion of $\mathbb{R}P^{2^r}$ in $\mathbb{R}^{2^r+k}$

then $w(\nu(\mathbb{R}P^{2^r})) = 1 + a + \dots + a^{2^r-1}$

so dim of fibers of $\mathbb{R}P^{2^r}$ is at least $2^r - 1$

but this is k

we are left to prove lemma 7

for this recall

$$\gamma_n = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in \ell\}$$

so $\gamma_n^\perp = \{ v \in \{x\} \times \mathbb{R}^{n+1} \mid v \perp (\gamma_n)_x \text{ for } x \in \mathbb{R}P^n \}$

now $w(\gamma_n^\perp) = (w(\gamma_n))^{-1} = (1+a)^{-1} = (1 + a + \dots + a^n)$

<u>lemma 9:</u>

$$\boxed{T\mathbb{R}P^n \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^\perp)}$$

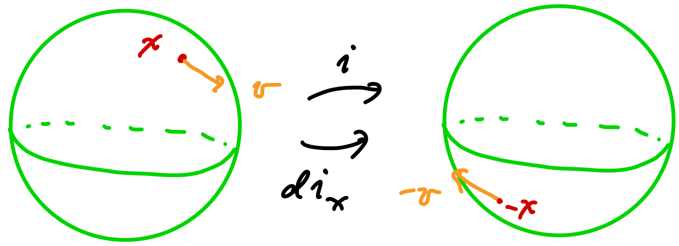
<u>Proof:</u> recall $\exists$ an involution $i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : x \mapsto -x$

and $\mathbb{R}P^n = S^n /_{x \sim i(x)}$

so $q : S^n \to \mathbb{R}P^n$ is a 2-fold cover with deck transform i

<u>note:</u> $dq_x(v) = dq_{-x}(-v)$

since $i(x) = -x$ and $di_x(v) = -v$



$\therefore$ tangent vectors to $\mathbb{RP}^n$ are the same as pairs $\{(x,v),(-x,-v)\}$ with

$$x\cdot x = 0 \quad \text{and} \quad x\cdot v = 0$$

let $l = \operatorname{span}(x)$

the pair above is determined by a linear map

$$f: L \to L^\perp$$

(given $v \in L^\perp = T_x S^n$ then $f: L \to L^\perp : x \to v$

get same map if you take $-v \in L^\perp = T_{-x} S^n$

and given $f: L \to L^\perp$ let x be a unit elt in L and $v = f(x)$ this gives a well-defined $\{(x,v),(-x,-v)\}$ in $T_{q(x)}\mathbb{RP}^n$)

so clearly $T_{q(x)}\mathbb{RP}^n \cong \operatorname{Hom}(\gamma_n, \gamma_n^\perp)$

<u>exercise</u>: Show this works on the level of bundles too

<u>Hint</u>: Consider local trivializations

<u>Proof of lemma 7</u>: from lemma 9 we see

$$T\mathbb{RP}^n \cong \mathrm{Hom}(\gamma_n, \gamma_n^\perp)$$

now note $\mathrm{Hom}(\gamma_n, \gamma_n)$ is a line bundle over $\mathbb{RP}^n$

and $\sigma : \mathbb{RP}^n \to \mathrm{Hom}(\gamma_n, \gamma_n) : x \mapsto id : (\gamma_n)_x \to (\gamma_n)_x$

is a non zero section

exercise: So $\mathrm{Hom}(\gamma_n, \gamma_n) \cong \mathbb{RP}^n \times \mathbb{R}$

(in general a rank n vector bundle E over M is trivial $\Leftrightarrow \exists$ n sections $s_1, \dots, s_n$ s.t. $s_1(x), \dots, s_n(x)$ spans E_x for all $x \in M$)

exercise: $\mathrm{Hom}(\gamma_n, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n, \gamma_n) \cong \mathrm{Hom}(\gamma_n, \gamma_n^\perp \oplus \gamma_n)$

$$\therefore \mathrm{Hom}(\gamma_n, \gamma_n^\perp) \oplus \underbrace{\varepsilon_1}_{\text{trivial rank 1 bundle}} = \mathrm{Hom}(\gamma_n, \underbrace{\gamma_n^\perp \oplus \gamma_n}_{\text{trivial rank } n+1 \text{ bundle}})$$

exercise:

1) $\mathrm{Hom}(\gamma_n, \varepsilon^{n+1}) \cong \mathrm{Hom}(\gamma_n, \varepsilon^1) \oplus \dots \oplus \mathrm{Hom}(\gamma_n, \varepsilon^1)$

2) $\mathrm{Hom}(\gamma_n, \varepsilon^1) \cong \gamma_n$

Hint: fix a metric g on γ_n

define $\phi_g : \gamma_n \to \mathrm{Hom}(\gamma_n, \varepsilon^1)$
$v \mapsto g(v, \cdot)$

thus $T\mathbb{RP}^n \cong \underbrace{\gamma_n \oplus \dots \oplus \gamma_n}_{n+1 \text{ times}}$

and property 3) in Th$^{\underline{m}}$ 5 gives

$$w(T\mathbb{R}P^n) = \left(w(\gamma_n)\right)^{n+1} = (1+a)^{n+1}$$

Bounding Manifolds (all manifolds connected)

given a closed manifold M of dimension n

and non-negative integers $i_1 \dots i_k$ s.t. $i_1 + \dots + i_k = n$

then $w_{i_1}(TM) \cup \dots \cup w_{i_k}(TM)$ is in $H^n(M; \mathbb{Z}/2)$

recall from algebraic topology $H_n(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$

let $[M]$ be a generator (called fundamental class)

we call $w_{i_1}(TM) \cup \dots \cup T_{i_k}(TM)\,([M]) \in \mathbb{Z}/2$

a Steifel-Whitney number of M

exercise: the Steifel-Whitney numbers of

$\mathbb{R}P^2$ are 1 and 0

$\mathbb{R}P^3$ are 0

$\mathbb{R}P^n$ are ?

Thm 10: ← Pontrjagin

> If M is the boundary of a compact manifold W
> then all the Steifel-Whitney numbers are zero

example: So $\mathbb{R}P^2$ does not bound any compact 3-manifold (oriented or not)!

and if Σ is any oriented surface since

$\Sigma = \partial M^3$ we have all Steifel-Whitney numbers are zero

Proof: recall we have a fundamental class

$$[W] \in H_{n+1}(W, M) \quad \text{and } [M] \in H_n(M)$$

moreover in L.E.S. of pair

$$\begin{aligned} H_{n+1}(W,M) &\xrightarrow{\partial} H_n(M) \\ [W] &\longmapsto [M] \end{aligned}$$

and in cohomology we have

$$H^n(M) \xrightarrow{\delta} H^{n+1}(W,M)$$

and for $\alpha \in H^n(M)$ generalize Stokes Th$^{\underline{m}}$

$$\delta\alpha[h] = \alpha(\partial[h]) \quad \forall [h] \in H_{n+1}(W,M)$$

now fixing a Riemannian metric on W we see
$\nu(M) = M \times \mathbb{R} = \varepsilon^1_M$

so $TW|_M \cong TM \oplus \varepsilon^1_M$

so $w(TW|_M) = w(TM)$

i.e. $i^*(w(TW)) = w(TM)$ $\quad$ $i: M \to W$ inclusion

the L.E.S. of a pair gives

$$H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W,M)$$

so for any $w_{i_1} \dots w_{i_k}$ s.t. $w_{i_1} \cup \dots \cup w_{i_k} \in H^n(W)$ we have

$$\delta(w_{i_1} \cup \dots \cup w_{i_k}) = 0$$

$$\therefore \quad w_{i_1} \cup \dots \cup w_{i_k}([M]) = w_{i_1} \cup \dots \cup w_{i_k}(\partial[W])$$
$$= \delta(w_{i_1} \cup \dots \cup w_{i_k})([W]) = 0$$

so all Steifel-Whitney numbers are 0 $\blacksquare$

Fact (Thom):

> If all Steifel-Whitney numbers of M are 0 then M is the boundary of some compact smooth manifold!

given 2 unoriented manifolds M_1 and M_2 we say they are unoriented cobordant if $\exists$ a compact manifold W with $\partial W = M_1 \cup M_2$

Corollary of Fact:

> M_1 and M_2 are unoriented cobordant
>
> $\Longleftrightarrow$
>
> they have the same Steifel-Whitney numbers

Proof of Th$^{\underline{m}}$ 5: recall we only need to check item 3) and uniqueness

We start with uniqueness:

recall we have the universal line bundle γ_n over $\mathbb{R}P^n$

$$\gamma_n = \{ (\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in \ell \}$$

and γ over $\mathbb{R}P^\infty$

note 4) and 2) determine $w_i(\gamma) = \begin{cases} 1 & i=0 \\ a & i=1 \\ 0 & i>1 \end{cases}$

in the next subsection we will show:

if $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a line bundle over M a paracompact space

then $\exists$ a map $f: M \to \mathbb{R}P^\infty$

such that $f^*(\gamma) = E$

(we will show much more!)

$\therefore$ 1) (and 2), 4)) $\Rightarrow$ we know $w(E)$ for <u>any</u> line bundle!

we can now finish uniqueness by

lemma 11 (Splitting Principle):

given $\begin{matrix} E \\ \downarrow p \\ M \end{matrix}$ there exists a space A and map $f: A \to M$ s.t.

1) f^*E is a direct sum of line bundles

2) $f^*: H^*(M) \to H^*(A)$ is injective

since $w(E)$ is determined by $w(f^*(E))$
and $w(f^*(E))$ is determined by 1)–4)
we are done with uniqueness!

exercise: 1) show $w(E \oplus E') = w(E) \cup w(E)$ is true for line bundles

2) show this + lemma 11 $\Rightarrow$ 3) in Th^m 5

Proof of lemma 11:

induct on the dimension of the fiber of $p: E \to M$

dim $= 1$: nothing to show

dim $= 2$: let $P(E) =$ projective bundle of E, $f: P(E) \to M$

i.e. in each fiber of E replace $\mathbb{R}^2$ with $\mathbb{R}P^1$

note: the transition maps $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(2;\mathbb{R})$ act on $\mathbb{R}P^1$ too

so to E there is an associated $\mathbb{R}P^1$-bundle

$$P(E) = \{\text{lines in } E_x \mid \forall x \in M)$$

so
$$\begin{array}{ccc} S^1 & \to & P(E) \\ & & \downarrow \\ & & M \end{array}$$

now consider $f^* E = \{(\ell, v) \in P(E) \times E : \pi(\ell) = p(e)\}$

let $\gamma_1 = \{ (\ell, v) \in P(E) \times E : \begin{matrix} \pi(\ell) = p(E) \\ v \in \ell \end{matrix} \}$

exercise: this is a line bundle over $P(E)$

$\gamma_1^{\perp} = \{ (\ell, v) \in f^*E : v \perp \ell \}$

so $\pi^* E \cong \gamma_1 \oplus \gamma_1^{\perp}$

later we will see (maybe) $H^*(P(E))$ is a free $H^*(M)$-module via $f^*(\alpha) \cup$.

$$f^* : H^*(M) \longrightarrow H^*(P(E))$$

is injective

now for $n > 2$ let $p : E \to M$ be an $\mathbb{R}^n$-bundle an

$f : P(E) \to M$ be $\mathbb{R}P^{n-1}$-bundle over M

then as above $f^*E = \gamma \oplus \gamma^{\perp}$

but now by induction $\exists\ \bar{f} : A \to P(E)$

s.t. $\bar{f}^* \gamma^{\perp} = \oplus$ line bundles (by induction)

and $\bar{f}^* : H^*(M) \to H^*(A)$ injective

so $(\bar{f} \circ f)^* E = \oplus$ line bundles

and $(\bar{f} \circ f)^* : H^*(M) \to H^*(A)$ injective

Remark: so we have seen Steifel-Whitney classes can be used to

1) obstruct k-frames over varios skeleta

2) say when we can reduce the structure group

e.g. $w_1(E)=0 \Leftrightarrow$ structure group reduces from $O(n)$ to $SO(n)$

3) differentiate bundles

4) obstruct immersions and embeddings

5) obstruct bounding a compact mfd.

there are many more applications and similarly for $c_i(E)$ and $p_i(E)$

example:

if Pontrjagin numbers not 0 then

1) no orientation reversing diffeomorphism and

2) does not bound an compact oriented mfd

e.g. $\mathbb{C}P^{2n}$ has no or$^{\underline{n}}$ reversing diffeo. and does not bound an oriented mfd

note: can prove both these with intersection pairings (e.g. Poincaré duality)

C. Classifying Spaces

Grassmannian

Recall $G_{n,m}$ = all n-dim'l subspaces in $\mathbb{R}^m$

we have $G_{n,m} \subset G_{n,m+1}$

so $G_n = \bigcup_m G_{n,m}$